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Rational solutions of fifth-order evolutionary equations for describing waves on water☆

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ABSTRACT

A method is proposed for obtaining the exact solutions of evolutionary equations in the form of a rational function. Invariant manifolds of the equations are used which have the same form of dependence on the required function and its derivatives as the generalized Riccati equations. Using fifth-order Kawahara and Korteweg–de Vries equations as an example, it is shown that their known particular solutions can be obtained using this method. New solutions of a non-linear fifth-order equation, which is encountered when describing long waves on water, are obtained.

1. Rational solutions

Particular solutions of the evolutionary equations (EEs)

$$u_t = F[u] \tag{1.1}$$

where $F[u] \equiv F(x, u, u_x, \ldots, \partial^m u/\partial x^m)$ is a certain smooth function, often have the form

$$u(t, x) = \frac{\varphi_0(t)f_0(x) + \varphi_1(t)f_1(x) + \ldots + \varphi_n(t)f_n(x)}{\varphi_0(t)g_0(x) + \varphi_1(t)g_1(x) + \ldots + \varphi_n(t)g_n(x)}, \quad n \geq 1 \tag{1.2}$$

Unlike the "polynomial" solutions

$$u(t, x) = \varphi_1(t)f_1(x) + \varphi_2(t)f_2(t) + \ldots + \varphi_n(t)f_n(x) \tag{1.3}$$

which have been studied previously,[1–3] we shall, for brevity, refer to solutions (1.2) as rational solutions of EEs (1.1).

A general approach to the construction of such solutions is proposed, which is based on the method of differential constraints[4] (or invariant manifolds[5]) which are sought in the form of generalized Riccati equations.[6,7]

For $n \leq 6$, this approach is illustrated using the example of the fifth order Korteweg - de Vries (KdV) equation (for the derivation, see Ref 8, for example)

$$u_t = u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x + \alpha(u_{xxx} + 6uu_x) \tag{1.4}$$

the equations of long waves in a shallow water under an ice sheet[9]

$$u_t + uu_x + u_{xxx} = u_{xxxxx} \tag{1.5}$$

and the previously studied[10] equations describing long waves on water

$$u_t + u_{xxxxx} - \alpha uu_{xxx} + 2(\alpha - 30)u_x u_{xx} + u_{xxx} - \beta uu_x = 0, \quad \alpha, \beta \neq 0 \tag{1.6}$$

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2. Invariant manifolds

The ordinary differential equation (ODE)

$$\Phi[u] \equiv \Phi(x, u, u_x, \ldots, \partial^n u/\partial x^n) = 0 \tag{2.1}$$

defines an invariant manifold[5] of EEs (1.1), or, in common terms, a differential constraint of EEs (1.1), if the relation

$$D_t\Phi[u]\big|_{[\Phi]} = 0; \quad D_t = \partial_t + F\partial_u + D_xF\partial_{u_x} + D_x^2F\partial_{u_{xx}} + \ldots \tag{2.2}$$

is satisfied. Here, D_t is the operator for the total derivative with respect to t by virtue of EEs (1.1), D_x is the operator for the total derivative with respect to x, and $[\Phi]$ denotes ODE (2.1) and its differential implications of the first, . . ., and m-th order. Since the operators $F[u]$ in EEs (1.1) and $\Phi[u]$ in ODE (2.1) do not explicitly contain t, the operators $F[y]$ and $\Phi[y]$ for the function $y(x)$ will be considered together with them.

Theorem 1. *Every ODE*

$$\Phi[y] \equiv \Phi(x, y, y', \ldots, y^{(n)}) = 0, \tag{2.3}$$

defining an invariant manifold of EEs (1.1) is invariant under the Lie–Becklund operator

$$X = F(x, y, y', \ldots, y^{(m)})\partial_y \tag{2.4}$$

with a coordinate $F[y]$ defined by the right-hand side of EEs (1.1).

For the proof, it is sufficient to note that relation (2.2) is identical with the invariance criterion[11]

$$\tilde{X}\Phi[y]\big|_{[\Phi]} \equiv F\Phi_y + D_xF\Phi_{y'} + \ldots + D_x^nF\Phi_{y^{(n)}}\big|_{[\Phi]} = 0 \tag{2.5}$$

of Eq. (2.3) with respect to operator (2.4), where

$$\tilde{X} = F\partial_y + D_xF\partial_{y'} + D_x^2F\partial_{y''} + \ldots \tag{2.6}$$

Relation (2.5) is an identity with respect to the variables $x, y, y', \ldots, y^{(n-1)}$. All the invariants of ODE (2.3) with respect to operator (2.4) can be found for each n if the dependence $\Phi[y]$ on $y, y', \ldots, y^{(n)}$ is specified with undetermined coefficients that are functions of x. In this case, the decomposition of equality (2.5) with respect to $y, y', \ldots, y^{(n-1)}$ leads to a system of governing equations in these coefficients.

Suppose an invariant manifold, giving the ODE (2.3), is found and its solution $y(x) = Y(x, c_1, \ldots, c_n)$ is known. Substitution of this solution in the form

$$u(t, x) = Y(x, \varphi_1(t), \ldots, \varphi_n(t))$$

into EEs (1.1) then leads to a system of ODEs in the functions $\varphi_1(t), \ldots, \varphi_n(t)$. Hence, the polynomial solutions (1.3) are generated by ODEs (2.3) that are linear in $y, y', \ldots, y^{(n)}$ with a general solution[1–3]

$$y(x) = c_1f_1(x) + c_2f_2(x) + \ldots + c_nf_n(x)$$

Hence, in order to construct the rational solutions (1.2) of EEs (1.1), it is necessary to determine the class of ODEs (2.3), the general solution of which has the form

$$y(x) = \frac{c_0f_0(x) + c_1f_1(x) + \ldots + c_nf_n(x)}{c_0g_0(x) + c_1g_1(x) + \ldots + c_ng_n(x)}, \quad n \geq 1 \tag{2.7}$$

3. Generalized Riccati equations

Vessiot[6] introduced high order equations into the treatment, possessing a common solution in the form of the ratio (2.7). An n-th order Riccati equation is obtained by eliminating the parameters c_i from equalities (2.7), and, with certain coefficients, $a_i = a_i(x)$ it has the form

$$n = 1, \quad R_1[y] \equiv a_0y' + a_1y^2 + a_2y + a_3 = 0 \tag{3.1}$$

$$n = 2, \quad R_2[y] \equiv a_0(yy'' - 2y'^2) + a_1y'' + a_2yy' + a_3y' + a_4y^3 + a_5y^2 + a_6y + a_7 = 0 \tag{3.2}$$

$$n=3,\quad R_3[y]\equiv a_0(2y'y'''-3y''^2)+a_1(y^2y'''-6yy'y''+6y'^3)+$$
$$+a_2(yy'''-3y'y'')+a_3y'''+a_4(y^2y''-2yy'^2)+a_5yy''+a_6y''+$$
$$+a_7y'^2+a_8y^2y'+a_9yy'+a_{10}y'+a_{11}y^4+a_{12}y^3+a_{13}y^2+a_{14}y+a_{15}\;=\;0,\;\ldots \qquad (3.3)$$

The Riccati equations $R_n[y]=0$, $n\geq 2$ possess the same properties as the Riccati equation (3.1). However, using the differential substitution, they reduce to an $(n+1)$-order linear equation[6] and their general solution can be obtained without quadratures if $2n+1$ particular solutions are known.[7]

Comparing the number of coefficients $f_i(x)$ and $g_i(x)$ in equalities (2.7) and the number of parameters $a_i(x)$ in Eqs. (3.2) and (3.3), we see that the parameters of the equations $R_2[y]=0$ are associated by two relations and the equations $R_3[y]=0$ are associated by eight relations and so on. They are well known in the case of Eq. (3.2).[6] If $a_0\neq 0$, then the equation $R_n[y]=0$ is reduced by means of the substitution $y=1/\bar{y}-a_1/a_0$ to an equation of the same form with $a_0=0$, $a_1=1$ which is a solution of the Riccati equation if and only if

$$9a_4\;=\;a_2^2,\quad 3a_5\;=\;a_2'+a_2a_3$$

The equations $R_n[y]=0$ (the second Painlevé equation, for example) have the same form of dependence on $y, y', \ldots, y^{(n)}$ as (3.2), (3.3) and so on but, not being Riccati equations, they often possess one-parameter families of solutions

$$y(x)\;=\;\frac{f_0(x)+c_1f_1(x)+c_1^2f_2(x)+\ldots+c_1^kf_k(x)}{g_0(x)+c_1g_1(x)+c_1^2g_2(x)+\ldots+c_1^kg_k(x)},\quad k\geq 1 \qquad (3.4)$$

In the majority of cases, they correspond to self-similar solutions of EEs (1.1) or solutions of a travelling wave type. The function (3.4) is the general solution of a first-order Riccati equation of degree k

$$k\;=\;2,\quad R_1^2[y]\equiv R_{02}^2-R_{01}R_{12}\;=\;0$$
$$k\;=\;3,\quad R_1^3[y]\equiv R_{03}^3+2R_{12}R_{03}^2-3R_{13}R_{02}R_{03}+R_{23}R_{02}^2+(R_{13}^2-R_{12}R_{23})R_{01}\;=\;0,\;\ldots$$

where $R_{ij}=0$ is the ordinary Riccati equation with the general solution

$$y(x)\;=\;(f_i(x)+cf_j(x))/(g_i(x)+cg_j(x)),\quad i,j\;=\;0,\ldots,k$$

This means that, with certain coefficients $b_i=b_i(x)$, the equations

$$R_1^k[y]\;=\;0 \qquad (3.5)$$

have the form

$$k\;=\;2,\quad R_1^2[y]\equiv y'^2+y'\sum_2(y)+\sum_4(y)\;=\;0$$
$$k\;=\;3,\quad R_1^3[y]\equiv y'^3+y'^{12}\sum_2(y)+y'\sum_4(y)+\sum_6(y)\;=\;0$$

Here,

$$\sum_2(y)=b_0y^2+b_1y+b_2,\;\sum_4(y)=b_3y^4+b_4y^3+\ldots\;+b_7,\;\sum_6(y)=b_8y^6+b_9y^5+\ldots+b_{14},\;\ldots$$

The determination of the invariant manifolds defined by Eq. (3.5) is a laborious problem. All of the non-linear invariant manifolds of the form

$$R_n[y]\;=\;0,\quad n\leq 6 \qquad (3.6)$$

are therefore first found for EEs (1.4)–(1.6).

The extended operator (2.6) is compiled using the right-hand side $F[y]$ of the EE. It acts on Eq. (3.6) with undetermined coefficients $a_i(x)$. After the derivatives $y^{(n)}, \ldots, y^{(n+m)}$ are substituted into the resulting equality, it is decomposed with respect to $y, y', \ldots, y^{(n-1)}$ in view of Eq. (3.6) and $D_x^j R_n[y] = 0 (j = 1, \ldots, m)$. This gives an overdetermined system of ODEs in the functions $a_i(x)$. Obviously, non-integrable m-th order EEs can only have non-linear invariant manifolds of the form (3.6) when $n \le m$.

The particular solutions (3.4) of ODEs (3.6) that have been found in this manner enable us to construct certain rational solutions of Eqs. (1.4)–(1.6). In many cases, this is only possible in the case of particular values of the parameters of ODEs (3.6). For brevity, those parameters that have an influence on the final form of the solution of the EEs are presented in this paper. Thus, in the case of travelling wave-type solutions of Eqs. (1.5) and (1.6), the parameter K_1 is equal to the velocity of the wave.

The functions (3.4) can be sought as solutions of the first order ODEs (3.5), $k \le n$, which are the integrals of Eq. (3.6). The coefficients $b_i(x)$ of the integral (3.5) are found from the overdetermined system of ODEs. It is obtained accompanying the decomposition of (3.6) with respect to y and y' in which the derivatives $y'', \ldots, y^{(n)}$ and, then, $(y')^j$ $(j \ge k)$ are replaced in view of Eq. (3.5) and its differential corollaries.

4. The fifth order Korteweg–de Vries (KdV) equation

The Lie–Becklund operator

$$X = F[y]\partial_y, \quad F[y] = y^V + 10yy''' + 20y'y'' + 30y^2y' + \alpha(y''' + 6yy') \tag{4.1}$$

corresponds to EE (1.4).

The even-order equations

$$R_{2n}[y] \equiv P_n + \sum_{i=0}^{n} K_i P_{i-1} = 0, \quad K_i = \text{const}, \quad n \in N \tag{4.2}$$

which define an infinite sequence of invariant manifolds of the KdV equation,[12] are invariant under to the operator (4.1) and their differential corollaries $R_{2n+1}[y] \equiv D_x R_{2n}[y] = 0$. The operator P_i in Eq. (4.2) is found using the Lenard formula

$$P_i' = P_{i-1}''' + 4yP_{i-1}' + 2y'P_{i-1}, \quad P_{-1} = 1/2$$

Eq. (4.2) when $n = 1$

$$R_2[y] \equiv y'' + 3y^2 + K_1 y + K_0/2 = 0 \tag{4.3}$$

has the integral

$$y'^2 + 2y^3 + K_1 y^2 + K_0 y + C_1 = 0$$

and, consequently, its general solution is expressed in terms of elliptic functions. When $K_0 = K_1^2/6 - 8\gamma^4/3$, Eq. (4.3) has particular solutions with $m = 2$ and $M = -K_1/6$

$$y(x, c_1) = m\gamma^2\left(\frac{2}{1 \pm \operatorname{ch} 2\gamma(x + c_1)} - \frac{1}{3}\right) + M$$

$$y(x, c_1) = m\gamma^2\left(\frac{1}{3} - \frac{1}{\cos^2 \gamma(x + c_1)}\right) + M$$

$$y(x, c_1) = M - \frac{m}{(x + c_1)^2}, \quad \gamma = 0 \tag{4.4}$$

Substitution of the solutions $u(t, x) = y(x, \varphi(t))$ into Eq. (1.1) leads to the relation

$$\varphi'(t) = \frac{8}{3}\gamma^4 + \frac{5}{6}K_1^2 - \alpha K_1$$

and, in particular, the single-soliton solution of Eq. (1.4) is obtained from this

$$u(t, x) = \frac{2\gamma^2}{\operatorname{ch}^2 \gamma\left(x + \left(\frac{8}{3}\gamma^4 + \frac{5}{6}K_1^2 - \alpha K_1\right)(t + t_0)\right)} - \frac{2}{3}\gamma^2 - \frac{1}{6}K_1$$

The general solution of Eq. (4.2) when $n = 2$

$$R_4[y] \equiv y^{IV} + 10yy'' + 5y'^2 + 10y^3 + K_2(y'' + 3y^2) + K_1 y + K_0/2 = 0 \tag{4.5}$$

is expressed in terms of hyperelliptic functions.[13–15] If $K_0 = K_2^3/50$ and $K_1 = 3K_2^2/10$, then Eq. (4.5) has the integral

$$(y' + 4/x^3)^2 + 2(y - 2/x^2 + K_2/10)^3 = 0$$

By means of the substitution

$$y = 2x^{-2} - 2z^2 - K_2/10$$

it is transformed into the Riccati equation $z' + z^2 - 2x^{-2} = 0$, from which it is possible to obtain the particular solution of Eq. (4.5)

$$y(x, c_1) = 6x(2c_1 - x^3)/(x^3 + c_1)^2 - K_2/10$$

Eq. (4.5) is self-similar and the function $y(x + c_2, c_1)$ is therefore also its solution. Substitution of the function $u(t, x) = y(x + \varphi_2(t), \varphi_1(t))$ into Eq. (1.4) leads to the relations

$$\varphi_1'(t) = 12(K_2 - \alpha), \quad \varphi_2'(t) = 3K_2(K_2 - 2\alpha)/10$$

which, when $K_2 = 0$, gives the well-known self-similar solution of the KdV equation

$$u_t = u_{xxx} + 6uu_x$$

which, at the same time, will not be self-similar in the case of Eq. (1.4).

Elimination of y''' from the integrals of the third-order Eq. (4.5)

$$2y'y''' - y''^2 + 10yy'^2 + 5y^4 + K_2(y'^2 + 2y^3) + K_1y^2 + K_0y = C_1$$

$$y'''^2 + 12yy'y''' + 4yy''^2 - 2y'^2y'' + 20y^3y'' + 30y^2y'^2 + 24y^5 +$$
$$+ K_2(y'' + 3y^2)^2 + K_1(2yy'' - y'^2 + 4y^3) + K_0(y'' + 3y^2) = C_2$$

leads to a second-order equation of the fourth degree in y''. Its integration when

$$C_1 = C_2 = K_0 = 0, \quad K_1 = 16m^2n^2, \quad K_2 = \mp 4(m^2 + n^2)$$

enables us to obtain the two-parameter families of solutions of Eq. (4.5)

$$y(x, c_1, c_2) = 2(\ln f_\pm)_{xx}$$

$$f_+ = 1 + e^{2m(x + c_1)} + e^{2n(x + c_2)} + \frac{(m - n)^2}{(m + n)^2}e^{2m(x + c_1)}e^{2n(x + c_2)}$$

$$f_- = n\sin m(x + c_1)\cos n(x + c_2) - m\sin n(x + c_2)\cos m(x + c_1)$$

From Eq. (1.4), where $u(t, x) = y(x, \varphi_1(t), \varphi_2(t))$, it follows that

$$\varphi_1'(t) = 4m^2(4m^2 \pm \alpha), \quad \varphi_2'(t) = 4n^2(4n^2 \pm \alpha)$$

and, in particular, the well-known two-soliton solution of Eq. (1.4) is obtained from this.

Eq. (4.2) when $n = 3$

$$R_6[y] \equiv y^{VI} + 7(2yy^{IV} + 4y'y''' + 3y''^2 + 10y^2y'' + 10yy'^2 +$$
$$+ 5y^4) + K_3(y^{IV} + 10yy'' + 5y'^2 + 10y^3) + K_2(y'' + 3y^2) + K_1y + K_0/2 = 0 +$$

if

$$K_0 = \frac{5}{2744}K_3^4, \quad K_1 = \frac{5}{98}K_3^3, \quad K_2 = \frac{5}{14}K_3^2$$

has the particular solutions

$$y_1(x, c_1) = -6\frac{x(2x^9 + 75c_1^2x^3 + 50c_1^3)}{(x^6 + 5c_1x^3 - 5c_1^2)^2} - \frac{K_3}{14}, \quad y_2(x, c_1) = -12\frac{x^{10} - 18c_1x^5 + 6c_1^2}{x^2(x^5 + 6c_1)^2} - \frac{K_3}{14}$$

One of them corresponds to the self-similar solution of the KdW equation but contradictory equalities are obtained when

$$u(t, x) = y_1(x + \varphi_2(t), \varphi_1(t))$$

is substituted into the fifth-order KdV equation. The other solution generates the self-similar solution of Eq. (1.4)

$$u(t, x) = y_2(x + \varphi_2(t), \varphi_1(t)), \quad \varphi_1'(t) = 120, \quad \varphi_2'(t) = -\frac{3}{10}\alpha^2, \quad K_3 = \frac{7}{5}\alpha$$

In addition to Eq. (4.2), the equations

$$R_2[y] \equiv y'' + y^2 + \frac{1}{5}\alpha y + K_0 = 0, \quad R_3[y] \equiv y''' + 6yy' + \frac{3}{5}\alpha y' - \frac{y'' + 3y^2 + \frac{3}{5}\alpha y}{x + K_1} = 0 \tag{4.6}$$

are invariant under to the operator (4.1) and their differential corollaries

$$R_3[y] \equiv y''' + \left(2y + \frac{1}{5}\alpha\right)y' = 0, \quad R_4[y] \equiv y^{IV} + 6(yy'' + y'^2) + \frac{3}{5}\alpha y'' = 0$$

The first of the particular solutions of the equation $R_2[y] = 0$ with $m = 6$ and $M = -\alpha/10$ generates a further single soliton solution of Eq. (1.4)

$$u(t, x) = \frac{6\gamma^2}{\text{ch}^2\gamma\left(x + \left(56\gamma^4 - \frac{3}{10}\alpha^2\right)(t + t_0)\right)} - 2\gamma^2 - \frac{1}{10}\alpha$$

Integration of the second equation of (4.6) leads to the ODE

$$y'' + 3y^2 + \frac{3}{5}\alpha y = 2C_1(x + K_1)$$

When $C_1 = 0$, this is a particular solution of Eq. (4.2), $n = 1$. If, however, $C_1 \neq 0$, then, by means of the substitution,

$$\bar{x} = -C_1^{-1/5}\left(x + K_1 + \frac{3}{100}\frac{\alpha^2}{C_1}\right), \quad \bar{y} = -\frac{1}{2}C_1^{-2/5}\left(y + \frac{1}{10}\alpha\right)$$

it is transformed into the first Painlevé equation $\bar{y}'' = 6\bar{y}^2 + \bar{x}$.

5. Kawahara equations

A Lie–Becklund operator (2.4) with the coordinate $F[y] = y^V - y''' - yy'$ corresponds to Eq. (1.5). The equation

$$R_4[y] \equiv y^{IV} - y'' - y^2/2 + K_1 y + K_0 = 0 \tag{5.1}$$

is invariant under this operator and its implication

$$R_5[y] \equiv y^V - y''' - yy' + K_1 y' = 0$$

All the rational solutions of Eq. (1.5) are therefore travelling wave-type solutions.

$$u(t, x) = y(x - K_1(t + t_0))$$

where $y(x)$ is the solution of Eq. (5.1).

Eq. (5.1) has the integral (compare with Ref. 9)

$$\left(y'^2 - \frac{4}{13}(y-C_1)(y-C_1+C_2)\right)^2 - \frac{16}{105}\left(y-C_1+\frac{5}{4}C_2\right)^2(y-C_1)^3 = 0$$

$$C_1, C_2 = \text{const}, \quad K_1 = 36/169 + C_1 - C_2/2 \qquad (5.2)$$

By means of the substitution $y = z^2 + C_1$, it reduces to the equation

$$z'^2 = \frac{1}{13}(z^2+C_2) \pm \frac{z}{\sqrt{105}}\left(z^2+\frac{5}{4}C_2\right)$$

the solution of which is expressed in terms of elliptic functions. Integration of Eq. (5.2) when $C_2 = 0$ gives the well-known[9] solitary-wave-type solution

$$y(x) = \frac{105}{169}\operatorname{ch}^{-4}\frac{x}{\sqrt{52}} + C_1$$

6. Rational solutions of Eq. (1.6)

The equations

$$R_2[y] \equiv y'' - 6y^2 + \frac{12-\beta}{\alpha}y + K_0 = 0 \qquad (6.1)$$

$$R_4[y] \equiv y^{IV} - \alpha yy'' + \left(\frac{3}{2}\alpha - 30\right)y'^2 + y'' - \frac{1}{2}\beta y^2 - K_1 y + K_0 = 0 \qquad (6.2)$$

and their corollaries

$$R_3[y] \equiv D_x R_2[y] = 0, \quad R_5[y] \equiv D_x R_4[y] = 0$$

are invariant under the Lie–Becklund operator (2.4) with the coordinate

$$F[y] = y^V - \alpha yy''' + 2(\alpha-30)y'y'' + y''' - \beta yy'$$

for an arbitrary value of the parameter α. When $\alpha = 12$, $\beta = -12\gamma^2$, the equations

$$R_3[y] \equiv y''' - 12yy' + (1+\gamma^2)y' + \gamma(y'' - 6y^2 + (1+\gamma^2)y + K_0) = 0 \qquad (6.3)$$

and $R_4[y] \equiv D_x R_3[y] = 0$ are also invariant under the operator (2.4) as are the equations

$$R_2[y] \equiv y'' - 6y^2 + K_1\sin(x+K_2) + K_0 = 0 \qquad (6.4)$$

and its differential implications when $\alpha = \beta = 12$ and the equation

$$R_4[y] \equiv y^{IV} - 18yy'' - 9y'^2 + 24y^3 + \frac{9-\beta}{5}y'' + \frac{9(\beta-8)}{10}y^2 + \frac{(\beta-6)(\beta-12)}{100}y + K_0 = 0 \qquad (6.5)$$

and its corollary $R_5[y] \equiv D_x R_4[y] = 0$ when $\alpha = 10$.

Eq. (6.1) with arbitrary α has the integral

$$y'^2 = 4y^3 - 12M_0 y^2 - 2K_0 y + C_1; \quad M_0 = (1-\beta/12)/\alpha \qquad (6.6)$$

and its solution is expressed in terms of elliptic functions. In particular, if

$$K_0 = \frac{2}{3}\gamma^4 - 6M_0^2$$

then there are solutions (4.4) with $m = -1$, $M = M_0$ to which the solution of Eq. (1.6)

$$u(t,x) = y(x, \varphi(t)), \quad \varphi'(t) = 4\gamma^4(\alpha-12)/3 + \beta M_0$$

corresponds.

All the solutions $y(x)$ of Eq. (6.2) generate solutions of Eq. (1.6) of the travelling-wave type

$$u(t,x) = y(x - K_1(t+t_0))$$

For arbitrary α, relation (6.6) is the integral of Eq. (6.2). There are other integrals of the form $R_1^k[y] = 0 (k = 2, 4)$ in the case when $\alpha = 10$, 15, -48.

When $\alpha = 15$,

$$(y'^2 - 2(y - B_1)(y^2 + B_2 y + C_1))^2 - 4(y^2 + B_2 y + C_1)^2 = 0$$
$$B_1 = (\beta + 18)/270, \quad B_2 = 2(\beta - 9)/135 \tag{6.7}$$

is also an integral of Eq. (6.2).

If $C_1 < B_2^2/4$, then

$$y^2 + B_2 y + C_1 = (y - a_1)(y - a_2)$$

and, by means of the substitution

$$y = a_1 + (a_1 - a_2)(z^2 - 1)^{-1}$$

the ODE (6.7) is transformed to the ODE

$$z'^2 = \frac{1}{2}(z-1)^2(z+1)(a_1 z + a_2 - B_1(z+1))$$

Its integration enables one to obtain the following unrestricted solutions of Eq. (6.2) with $\alpha = 15$

$$y(x) = \frac{1}{15} - \frac{4}{3}\gamma^2 + M\cos^2\gamma x + \frac{\gamma^2}{\cos^2\gamma x}; \quad \beta = 180\gamma^2, \quad K_1 = 176\gamma^4 - 12\gamma^2(1+5M) \tag{6.8}$$

$$y(x) = \frac{1}{15} + \frac{4}{3}\gamma^2 + \frac{4M}{e^{2\gamma x}}(e^{2\gamma x} + m)^2 - \frac{4m\gamma^2 e^{2\gamma x}}{(e^{2\gamma x} + m)^2}$$
$$\beta = -180\gamma^2, \quad K_1 = 176\gamma^4 + 12\gamma^2(1 + 5mM) \tag{6.9}$$

If $\alpha = -48$, Eq. (6.2) has the following integrals: (6.6) and

$$(y'^2 - 4(y + 2C_2 + C_0)(y^2 + (2C_0 - 2C_1 - C_2)y + (2C_1 - C_0)(C_2 - C_0) - 71C_1^2))^2 +$$
$$+ \frac{16}{105}(348C_2 - \beta)(y + 2C_1 + C_0)^3(y - 3C_1 + C_0)^2 = 0 \tag{6.10}$$
$$C_1, C_2 = \text{const}, \quad C_0 = \frac{1 - 13C_2}{48}, \quad K_1 = \beta C_0 + 18000C_1^2 - 36C_2^2$$

$$(y' \mp 4\gamma(y + \Gamma_1))^2 - 4(y + \Gamma_1)^3 = 0; \quad \Gamma_1 = \frac{52\gamma^2 + 1}{48} \tag{6.11}$$

when $\beta = 288\gamma^2$ and, also,

$$(y' \pm 12\gamma(y + \Gamma_2))^2 - 4(y + \Gamma_2)(y + \Gamma_3)^2 = 0; \quad \Gamma_2 = \frac{380\gamma^2 + 1}{48}, \quad \Gamma_3 = \frac{860\gamma^2 + 1}{48} \tag{6.12}$$

when $\beta = -192\gamma^2$.

The ODE (6.11) is reduced, by means of the substitution

$$y = z^{-2} - \Gamma_1$$

to the linear ODE $z' \pm 2\gamma z - 1 = 0$ from which the solution of Eq. (6.2)

$$y(x) = \gamma^2(\text{th}\gamma x \pm 1)^2 - \Gamma_1, \quad K_1 = 6\gamma^2(1 - 44\gamma^2) \tag{6.13}$$

can easily be obtained.

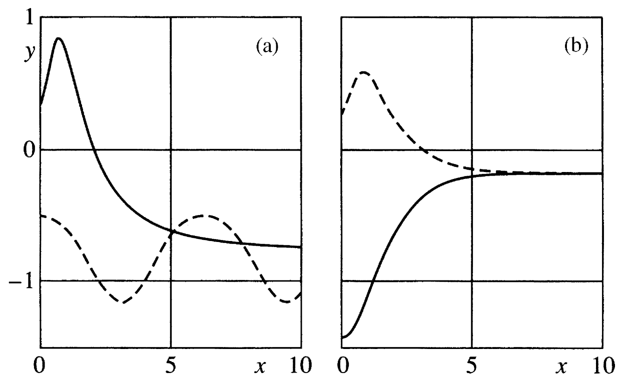

Fig. 1.

Eq. (6.12) is reduced by means of the substitution

$$y = z^2 - \Gamma_2$$

to the Riccati equation $z' \pm 6\gamma z = z^2 + 10\gamma$, which, in the case of Eq. (6.2) with $\alpha = -48$, gives the solution

$$y(x) = \gamma^2(\mathrm{tg}\gamma x \pm 3)^2 - \Gamma_2, \quad K_1 = 4\gamma^2(4\gamma^2 - 1) \tag{6.14}$$

Eq. (6.10) is reduced by the substitution

$$y = z^{-2} - 2C_1 - C_0$$

to the equation

$$z'^2 = (4C_1C_2 - 63C_1^2)z^4 + az(1 - 5C_1z^2) - (6C_1 + C_2)z^2 + 1, \quad 105a^2 = \beta - 348C_2$$

the solutions of which are expressed in terms of elliptic functions. For certain values of the parameters C_1 and C_2 it is possible to obtain the following solutions of Eq. (6.2)

$$y(x) = \frac{1680\beta}{(\beta x^2 - 420)^2} - \frac{1}{48}, \quad K_1 = \frac{\beta}{48} \tag{6.15}$$

$$y(x) = \frac{\beta m - 1}{48} + \frac{60\beta m^2(3m-5)(5\beta m(3m-5)x^2 - 9(108m^2 - 469m + 165))}{(5\beta m(3m-5)x^2 - 3(612m^2 - 1491m + 310))^2}$$

$$432(m^3 - m^2) + 468m - 155 = 0, \quad K_1 = \frac{\beta(1 - \beta m)}{48} \tag{6.16}$$

$$y(x) = \mp\frac{13\gamma^2 \pm 1}{48} + \frac{420\gamma^4}{(\beta \mp 72\gamma^2)(m + \tilde{y}_\pm(x))^2}$$

$$m^2 = \frac{\beta \pm 348\gamma^2}{\beta \mp 72\gamma^2}, \quad K_1 = \frac{\beta(1 \pm 13\gamma^2)}{48} - 36\gamma^4 \tag{6.17}$$

and, also, if $39m^4 - 3m^2 - 1 = 0$, then, when $\beta = \mp 108\gamma^2$,

$$y(x) = \mp\frac{10\gamma^2 \pm 1}{48} \mp \frac{35\gamma^2 m^3(2m + \tilde{y}_\pm(x))}{(4m^2 + 1)(m + \tilde{y}_\pm(x))^2}, \quad K_1 = \mp\frac{9}{4}\gamma^2(1 \pm 14\gamma^2) \tag{6.18}$$

Henceforth, only the upper or only the lower signs are taken; $\tilde{y}_+(x) = \mathrm{ch}\gamma x$, $\tilde{y}_-(x) = \cos\gamma x$.

The functions (6.15) and (6.16) are bounded when $\beta < 0$, and (6.17) is bounded when the lower sign is chosen, that is, when $\beta < -72\gamma^2$. The solution (6.18) is unbounded when the lower sign is chosen.

A graph of the solution (6.16) for $\beta = -84$ is shown by the continuous curve in Fig. 1, *a* and a graph of the solution (6.17) with the choice of the lower sign and $\gamma = 1$ is shown by the dashed curve. The function (6.18) with the choice of the upper sign and $\gamma^2 = 7/9$, $m > 0$ (the continuous curve) and $m < 0$ (the dashed curve) is shown in Fig. 1, *b*. All the solutions considered are symmetrical about the $x = 0$ axis and, therefore, only the domain $x \geq 0$ is shown in the figures.

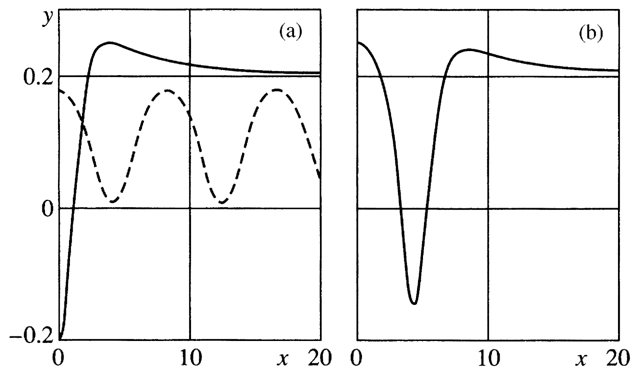

Fig. 2.

If $\alpha = 10$, then, in addition to the integral (6.6), Eq. (6.2) has an integral

$$\left(y'^2 - 4y^3 + \frac{12-\beta}{10}y^2 + C_1 y + C_1\frac{\beta-4}{40} - \frac{(\beta-4)^2}{2000}\right)^2 - 16C_2\left(y + \frac{\beta-4}{40}\right)^3 = 0 \qquad (6.19)$$

By means of the substitution

$$y = z^2 + (4-\beta)/40$$

it is reduced to the equation

$$z'^2 = z^4 - \frac{\beta}{20}z^2 \pm \sqrt{C_2}z - \frac{C_1}{4} + \frac{(\beta-4)(\beta+12)}{1600}$$

the solutions of which are expressed in terms of elliptic functions. The following solutions of Eq. (6.2) with $\alpha = 10$, $K_1 = \beta(\beta-6)/60 - 2\gamma^4/3$ are obtained from this for specific values of the parameters C_1 and C_2

$$y(x) = \frac{6-\beta}{60} \pm \frac{\gamma^2}{6} \mp \gamma^2\frac{1 + m\tilde{y}_\pm(x)}{(m + \tilde{y}_\pm(x))^2}, \quad m^2 = \frac{\beta \pm 20\gamma^2}{\beta \mp 10\gamma^2} \qquad (6.20)$$

$$y(x) = \frac{6-\beta}{60} + 2\beta\frac{\beta x^2 + 30}{(\beta x^2 - 30)^2}, \quad \gamma = 0 \qquad (6.21)$$

With the choice of the lower sign, the function (6.20) is bounded when $\beta < -10\gamma^2$, and (6.21) is bounded when $\beta < 0$. The graph of the solution (6.21) is represented by the continuous curve in Fig. 2, *a* for $\beta = -6$ and the graph of the solution (6.20) with the choice of the lower sign and $\gamma = 3/4$ is represented by the dashed curve.

After the transformation

$$y = z + (9-\beta)/90$$

Eq. (6.5), which defines the invariant manifold of Eq. (1.6) in the case when $\alpha = 10$, is identical to the self-similar equation F-VI from Ref. 14 in which fourth-and fifth-order ODEs with the Painlevé property were classified. Its general solution, like the solution of ODE (4.5), is expressed in terms of hyperelliptic functions. When $\beta = \pm 20(m^2 - 2n^2)$, there is a two parameter family of solutions

$$y(x, c_1, c_2) = \frac{1}{10} \pm \frac{1}{6}(4n^2 - m^2) - (\ln f_\pm)_{xx}$$

$$f_+ = (2(n^2 - m^2) + (m^2 + 2n^2)\operatorname{ch}m(x + c_1))\operatorname{ch}n(x + c_2) - 3mn\operatorname{sh}m(x + c_1)\operatorname{sh}n(x + c_2)$$

$$f_- = (2(n^2 - m^2) + (m^2 + 2n^2)\cos m(x + c_1))\cos n(x + c_2) + 3mn\sin m(x + c_1)\sin n(x + c_2)$$

to which the solution of Eq. (1.6)

$$u(t, x) = y(x, \varphi_1(t), \varphi_2(t)), \quad \varphi_1'(t) = \frac{\beta}{10} - \frac{\beta^2}{60} + \frac{2}{3}m^4, \quad \varphi_2'(t) = \frac{\beta}{10} - \frac{\beta^2}{120} - \frac{8}{3}n^4 \qquad (6.22)$$

corresponds.

The solution (6.22) is bounded when $\beta<0$ and $m<n$.

Eq. (6.5) has first order integrals: (6.6), (6.19) and

$$\left(y' - \frac{8}{(x+c_1)^3}\right)^2 - 4\left(y + \frac{4}{(x+c_1)^2} - \frac{\beta+12}{120}\right)\left(y - \frac{2}{(x+c_1)^2} + \frac{\beta-6}{60}\right)^2 = 0 \quad (6.23)$$

Eq. (6.23) is reduced by the substitution

$$y = z^2 - 4(x+c_1)^{-2} + (\beta+12)/120$$

to the Riccati equation $z' = z^2 - 6(x+c_1)^{-2} + \beta/40$, which is reduced by means of the substitution $z = v'/v$ to the linear ODE $v'' + (\beta/40 - 6(x+c_1)^{-2})v = 0$. When $\beta = \mp 40\gamma^2$,

$$y(x, c_1, c_2) = \frac{1}{10} \pm \frac{2}{3}\gamma^2 - (\ln f_\pm)_{xx}$$

$$f_+ = (3 + \gamma^2(x+c_1)^2)\,\mathrm{sh}\,\gamma(x+c_2) - 3\gamma(x+c_1)\,\mathrm{ch}\,\gamma(x+c_2)$$

$$f_- = (3 - \gamma^2(x+c_1)^2)\sin\gamma(x+c_2) - 3\gamma(x+c_1)\cos\gamma(x+c_2) \quad (6.24)$$

is a solution of Eq. (6.5). Substitution of the function $u(t, x) = y(x, \varphi_1(t), \varphi_2(t))$ into Eq. (1.6) leads to the relations

$$\varphi_1'(t) = -4\gamma^2(20\gamma^2 \pm 3)/3, \quad \varphi_2'(t) = -4\gamma^2(4\gamma^2 \pm 1)$$

This solution of Eq. (1.6) with $\alpha = 10$ is not self-similar nor a solution of the travelling wave type. When $\beta > 0$, the function (6.24) is unbounded. A graph of this function when $\beta = -6$ is shown in Fig. 2, *b*.

We will now consider the case when $\alpha = 12$. Integration of Eq. (6.3) leads to the ODE

$$y'' - 6y^2 + (1+\gamma^2)y + K_0 + 6\exp(-\gamma(x+c_1)) = 0$$

This equation, like (6.4), is not transformed to the first Painlevé equation. If

$$K_0 = -\frac{1}{24}\left(1 + \frac{5}{4}\gamma^2\right)\left(1 + \frac{3}{4}\gamma^2\right)$$

then it has the particular solution

$$y(x, c_1) = \exp\left(-\frac{1}{2}\gamma(x+c_1)\right) + \frac{1}{12}\left(1 + \frac{5}{4}\gamma^2\right)$$

The corresponding solution of Eq. (1.6) with $\alpha = 12$, $\beta = -12\gamma^2$ has the form

$$u(t, x) = y(x, \varphi(t)), \quad \varphi'(t) = -\gamma^2(1+\gamma^2)$$

If $K_0 = |K_1| + 1/384$, Eq. (6.4) has the particular solution $y(x)$, to which the stationary solution

$$u(t, x) = c_2\cos\frac{x+c_1}{2} - \frac{1}{48}$$

of Eq. (1.6) with $\alpha = 12$ and $\beta = 12$ corresponds.

7. Conclusion

Thus, a general approach to the construction of the exact solutions in the form (1.2), based on the use of invariant manifolds with the form of generalized Riccati equations has been proposed for the evolutionary Eq. (1.1). It has been shown, using the examples of Eqs. (1.4)–(1.6), that most of the known[8–10,12] exact solutions of these equations can be obtained by this method. The solutions of Eq. (1.6) with $\alpha = 15$, -48 and 10, corresponding to the functions (6.8), (6.9) (6.14) - (6.16), (6.18), (6.21) and (6.24) and, also, the two-soliton solution (6.22), are new.

Evolutionary equations can be subdivided into three classes with reference to the number of admissible invariant manifolds: 1) equations, all the rational solutions of which are contained among solutions of the travelling-wave type (such as the Kawahara and Kuramoto-Sivashinskii equations and Eq. (1.6) with arbitrary α, for example), 2) equations having other rational solutions, but the number of such equations is limited (Eq. (1.6) with $\alpha = 10$), and 3) integrable equations having an infinite number of invariant manifolds and, correspondingly, rational solutions (the KdV, Sawada–Kotera, and Kaup–Kuperschmidt equations)

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